Review: Principles of Mathematical Analysis

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Abstract

Here is the brief review of Rudin's "Principles of Mathematical Analysis". The main purpose of this document is to help myself (of course for other readers at the same time if there are some) touch basic concepts much clearly. To be brief, only important definitions and theorems would be listed, without exhaustive proofs. Enjoy the study !

Contents

1 The Real and Complex Number Systems

Introduction

1.3 Definition If A is any set (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a member (or an element) of A. If x is not a member of A, we write $x \notin A$.

The set which contains no element will be called the *empty set*. If a set has at least one element, it is called *nonempty*.

If A and B are sets, and if every element of A is an element of B, we say that A is a subset of B, and write $A \subset B$, or $B \supset A$. If, in addition, there is an element of B which is not in A, then A is said to be a *proper* subset of B. Note that $A \subset A$ for every set of A. If $A \subset B$ and $B \subset A$, we write $A = B$. Otherwise $A \neq B$.

Ordered Sets

1.5 Definition Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$ then one and only one of the statements is true.

$$x < y, \qquad x = y, \qquad y < x$$

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

1.6 Definition An *ordered set* is a set S in which an order is defined.

1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \subset S$ such that $x \leqslant \beta$ for every $x \in E$, we say that E is *bounded above*, and call β an *upper bound* of E. Lower bounds are defined in the same way with $\geqslant$.

1.8 Definition Suppose S is an ordered set, $E \in S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties :

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then α is called the *least upper bound of* E or the *supremum of* E, and we write

$$\alpha = \sup E$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner. The statement $\alpha = \inf E$ means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E.

1.10 Definition An ordered set S is said to have the *least-upper-bound property* if the following is true: If $E \subset S$, E is not empty, and E is bounded above, the $\sup E$ exists in S

1.11 Theorem Suppose S is an *ordered set with the least-upper-bound-property*, $B \subset S$, B is not empty, and B is *bounded below*. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.

Fields

1.17 Definition An *ordered field* is a *field* F which is also an *ordered set.*

The Real Field

1.19 Theorem There exists an *ordered field* R which has the least-upper-bound property. Moreover, R contains Q as a *subfield.*

If $x \in R$, $y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.

The Extended Real Number System

1.23 Definition The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R, and define $-\infty < x < +\infty$ for every $x \in R$.

The Complex Field

1.33 Theorem Let z and w be complex numbers, then (a) $|z| > 0$ *unless* $z = 0$, $|0| = 0$; (b) $|\overline{z}| = |z|$; (c) $|zw| = |z|\,|w|$; (d) $|\Re z| \leqslant |z|$; (e) $|z + w| \leqslant |z| + |w|$.

1.35 Theorem If $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \overline{b}_j\right|^2 \leqslant \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Euclidean Spaces

1.36 Definition ... we define the *norm* of $\mathbf{x}$ by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}$$

The structure now defined (the vector space R^k with the above inner product and norm) is called *euclidean k-space.*

2 Basic Topology

Finite, Countable, and Uncountable Sets

2.3 Definition If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 correspondence, or that A and B have the same *cardinal number*, or briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties: reflexive $A \sim A$; symmetric $A \sim B \Rightarrow B \sim A$; transitive $A \sim B$ and $B \sim C \Rightarrow A \sim C$. Any relation with these three properties is called an *equivalence relation*.

2.4 Definition For any positive integer n, let J_n be the set whose elements are the integers 1, 2, $\cdots, n$; let J be the set consisting of all positive integers. For any set A, we say

(a) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim J$.

(d) A is *uncountable* if A is neither finite nor countable.

(e) A is *at most countable* if A is finite or countable.

2.8 Theorem *Every infinite subset of a countable set A is countable.*

2.12 Theorem Let $\{E_n\}$, $n = 1, 2, 3, \cdots$, be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is *countable.*
Corollary Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable. Put

$$T = \bigcup_{\alpha \in A} B_\alpha$$

Then T is at most countable.

2.13 Theorem Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \cdots, a_n)$, where $a_k \in A$ $(k = 1, \cdots, n)$, and the elements $a_1, \cdots, a_n$ need not be distinct. Then B_n is countable.
Corollary The set of all rational numbers is countable.

2.14 Theorem Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is *uncountable.* # Turn nth digit of s_n to another value and there would be another novel sequence.

Metric Spaces

2.15 Definition A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p,q)$, called the *distance* from p to q, such that

(a) $d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$;

(b) $d(p,q) = d(q,p)$;

(c) $d(p,q) \leqslant d(p,r) + d(r,q)$ for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

2.17 Definition

If $a_i < b_i$ for $i = 1, \cdots, k$, the set of all points in R^k whose coordinates satisfy the inequalities $a_i \leqslant x_i \leqslant b_i$ $(1 \leqslant i \leqslant k)$ is called a *k-cell.*

If $\mathbf{x} \in R^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at $\mathbf{x}$ and radius $\mathbf{r}$ is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $\leqslant$).

We call a set $E \subset R^k$ *convex* if $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$, whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$.

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$, for some $r > 0$.

(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.

(c) If $p \in E$ and p is not a limit point of E, the p is called an *isolated point* of E.

(d) E is *closed* if every limit point of E is a point of E.

(e) A point p is an *interior* point of E if there is a neighborhood of p such that $N \subset E$.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(h) E is *perfert* if E is closed and if every point of E is a limit point of E.

(i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

(j) E is *dense* in X if every point of X is a limit point of E, or a point of E (or both).

2.19 Theorem Every neighborhood is an open set.
2.20 Theorem If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.
Corollary A finite point set has no limit points.

2.22 Theorem Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$$

2.23 Theorem A set E is open if and only if its complement is closed.
Corollary A set F is closed if and only if its complement is open.

2.24 Theorem

(a) For any collection $\{G_\alpha\}$ of open sets, $\cup_\alpha G_\alpha$ is open.

(b) For any collection $\{F_\alpha\}$ of closed sets, $\cap_\alpha F_\alpha$ is closed.

(c) For any finite collection $G_1, \cdots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.

(d) For any finite collection $F_1, \cdots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed.

2.26 Definition If X is a metric space, if $E \subset X$ and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\overline{E} = E \cup E'$.

2.27 Theorem If X is a metric space and $E \subset X$, then

(a) $\overline{E}$ is closed,

(b) $E = \overline{E}$ if and only if E is closed,

(c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

By (a) and (c), $\overline{E}$ is the *smallest* closed subset of X that contains E.

2.30 Theorem Suppose $Y \subset X$. A subset E of Y is *open relative to* Y if and only is $E = Y \cap G$ for some open subset G of X.

Compact Sets

2.31 Definition By an *open cover* of a set E in a metric space X, we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

2.32 Definition A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$

2.33 Theorem Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.
2.34 Theorem Compact subsets of metric spaces are closed.
2.35 Theorem Closed subsets of compact sets are compact.
Corollary If F is closed and K is compact, then $F \cap K$ is compact.

2.36 Theorem If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of *every finite* subcollection of $\{K_\alpha\}$ is nonempty, the $\cap K_\alpha$ is nonempty.
Corollary If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \cdots)$, then $\cap_1^\infty K_n$ is not empty.

2.37 Theorem If E is an infinite subset of a compact set K, then E has a limit point in K.

2.38 Theorem If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \cdots)$, then $\cap_1^\infty I_n$ is not empty.

2.39 Theorem Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \cdots)$, then $\cap_1^\infty I_n$ is not empty.

2.40 Theorem Every k-cell is compact.

2.41 Theorem If a set E in R^k has one of the following three properties, then it has the other two:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

2.42 Theorem (Weierstrass) Every bounded infinite subset of R^k has a limit point in R^k.
2.40+2.37

Perfect Sets

2.43 Theorem Let P be a nonempty perfect set in R^k. Then P is *uncountable.*
Corollary Every interval [a,b] (a<b) is uncountable. In particular, the set of all real numbers is uncountable.

The Cantor set, is clearly compact and nonempty. It provides us with an example of an uncountable set of measure zero.

Connected Sets

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A.

A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

2.46 Theorem A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

3 Numerical Sequences and Series

Convergent Sequences

3.1 Definition A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\epsilon > 0$ there is an integer N such that $n \geqslant N$ implies that $d(p_n, p) < \epsilon$.

Subsequences

3.5 Definition Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a *sequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

Cauchy Sequences

3.8 Definition A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geqslant N$ and $m \geqslant N$.

3.11 Theorem

(a) In any metric space X, every convergent sequence is a Cauchy sequence.

(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

(c) In R^k, every Cauchy sequence converges.

Note: The difference between the definition of convergence and the definition of a Cauchy sequence, is that the limit is explicitly involved in the former, but not in the latter.

3.12 Definition
A metric space in which every Cauchy sequence converges is said to be *complete*.

Upper and Lower Limits

3.19 Theorem If $s_n \leqslant t_n$ for $n \geqslant N$, where N is fixed, then

$$\lim_{n\to\infty} \inf s_n \leqslant \lim_{n\to\infty} \inf t_n$$
$$\lim_{n\to\infty} \sup s_n \leqslant \lim_{n\to\infty} \sup t_n$$

Some Special Sequences

3.20

(a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

(d) If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Series

3.22 Theorem $\sum a_n$ converges if and only if for every $\epsilon > 0$ there is an integer N such that

$$\left| \sum_{k=n}^{m} a_k \right| \leqslant \epsilon$$

if $m \geqslant n \geqslant N$.

In other words, we have 3.23:

3.23 Theorem If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

3.25 Theorem

(a) If $|a_n| \leqslant c_n$ for $n \geqslant N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.

(b) If $a_n \geqslant d_n \geqslant 0$ for $n \geqslant N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Series of Nonnegative Terms

3.26 If $0 \leqslant x \leqslant 1$, then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If $x \geqslant 1$, the series diverges.

3.28 $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leqslant 1$.

3.29 If $p > 1$,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \leqslant 1$, the series diverges.

The Number e

3.30 Definition

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

3.31 Theorem

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

And e is irrational.

The Root and Ratio Tests;

3.33 Theorem (Root Test) Given $\sum a_n$, put $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$. Then

(a) if $\alpha < 1$, $\sum a_n$ converges;

(b) if $\alpha > 1$, $\sum a_n$ diverges;

(c) if $\alpha = 1$, the test gives no information.

3.34 Theorem (Ratio Test) The series $\sum a_n$

(a) converges if $\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geqslant 1$ for all $n \geqslant n_0$, where n_0 is some fixed integer.

3.37 Theorem For any sequence $\{c_n\}$ of positive numbers,

$$\lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} \leqslant \lim_{n\to\infty} \inf \sqrt[n]{c_n}$$

$$\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leqslant \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$$

Power Series

3.38 Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

3.39 Theorem Given the power series $\sum c_n z^n$, put

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{c_n}, \qquad R = \frac{1}{\alpha}$$

Then the series converges if $|z| < R$, and diverger if $|z| > R$.

Summation by Parts

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Absolute Convergence

The series $\sum a_n$ is said to *converge absolutely* if the series $\sum |a_n|$ converges.
If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Addition and Multiplication of Series

3.50 Theorem Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely,

(b) $\sum_{n=0}^{\infty} a_n = A$,

(c) $\sum_{n=0}^{\infty} b_n = B$,

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n = 0, 1, 2, \cdots)$.

Then

$$\sum_{n=0}^{\infty} c_n = AB$$

That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

3.51 Theorem If the series $\sum a_n$, $\sum b_n$, $\sum c_n$ converge to A, B, C, and $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$. # Here no assumption is made concerning absolute convergence. We shall give a simple proof (which depends on the continuity of power series) after 8.2.

Rearrangements

3.52 Definition Let $\{k_n\}$, $n = 1, 2, 3, \cdots$, be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 functions from J onto J). Put

$$a'_n = a_{k_n} \qquad n = 1, 2, 3, \cdots$$

we say that $\sum a'_n$ is a *rearrangement* of $\sum a_n$.

3.54 Theorem Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose $-\infty \leqslant \alpha \leqslant \beta \leqslant +\infty$, then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that

$$\lim_{n\to\infty} \inf s'_n = \alpha \qquad \lim_{n\to\infty} \sup s'_n = \beta$$

3.55 Theorem If $\sum a_n$ is a series of complex numbers which converge absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

4 Continuity

Limits of Functions

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Continuous Functions

4.5 Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is said to be *continuous at* p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all points $x \in E$ for which $d_X(x, p) < \delta$. If f is continuous at every point of E, then f is said to be *continuous on* E.

4.8 Theorem A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.
Corollary A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

4.9 Theorem Let f and g be complex continuous functions on a metric space X. Then $f + g$, fg, and f/g are continuous on X. ($g(x) \neq 0$ for all $x \in X$ in the last case.)

4.10 Theorem Let $f_1, \cdots, f_k$ be real functions on a metric space X, and let $\mathbf{f}$ be the mapping of X into R^k defined by

$$\mathbf{f}(x) = (f_1(x), \cdots, f_k(x)) \qquad (x \in X)$$

then $\mathbf{f}$ is countinuous if and only if each of the functions $f_1, \cdots, f_k$ is coutinuous.

Continuity and Compactness

4.13 Definition A mapping $\mathbf{f}$ of a set E into R^k is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \leqslant M$ for all $x \in E$.

4.14 Theorem Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is *compact.* # By 4.8

4.15 Theorem If $\mathbf{f}$ is a continuous mapping of a compact metric space X into R^k, then $\mathbf{f}(x)$ is closed and bounded. Thus, $\mathbf{f}$ is bounded. # By 2.41

4.16 Theorem Suppose f is a continuous real function on a compact metric space X, and $M = \sup_{p \in X} f(p)$, $m = \inf_{p \in X} f(p)$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

4.18 Definition Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_Y(f(p), f(q)) < \epsilon$$

for all p and q in X for which $d_X(p, q) < \delta$.

First, uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point. Second, if f is continuous on X, then it is possible to find, for each

$\epsilon > 0$ and for each point p of X, a number $\delta > 0$ having the property specified in 4.5, in which δ depends on ϵ and p; however, if f uniformly continuous on X, then it is possible, for each $\epsilon > 0$, to find one number $\delta > 0$ which will do for all points p of X.

4.19 Theorem Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is *uniformly continuous* on X.

Continuity and Connectedness

4.22 Theorem If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected. # See 2.45 for *connected*.

Discontinuities

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is *discontinuous* at x, or that f has a discontinuity at x. If f is defined on an interval or on a segment, it is customary to divide discontinuities into two types.

4.26 Definition If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a *simple discontinuity* at x. Otherwise the discontinuity is said to be of the *second kind.*

Monotonic Function

4.28 Definition Let f be real on (a, b). Then f is said to be *monotonically increasing* on (a, b) if $a < x < y < b$ implies $f(x) \leqslant f(y)$.

Infinite Limits and Limits at Infinity

4.32 Definition For any real c, the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

5 Differentiation

The Derivative of a Real Function

5.1 Definition

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

If f is differentiable at a point x, then f is continuous at x.

Mean Value Theorems

5.9 Theorem If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x)$$

The Continuity of Derivatives

5.12 Theorem Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$. But f' may very well have discontinuities of the second kind.

L'Hospital's Rule

5.13 Theorem Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leqslant a < b \leqslant +\infty$. If $x \to a$, $f(x) \to 0$, $g(x) \to 0$ (or ∞), suppose as $x \to a$

$$\frac{f'(x)}{g'(x)} \to A \qquad \Rightarrow \qquad \frac{f(x)}{g(x)} \to A$$

Derivatives of Higher Order

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Taylor's Theorem

5.15 Theorem Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

Differentiation of Vector-valued Functions

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6 The Riemann-Stieltjes Integral

Definition and Existence of the Integral

6.1 Definition Let $[a,b]$ be a given interval. By a *partition* P of $[a,b]$ we mean a finite set of points $x_0, x_1, \cdots, x_n$, where

$$a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$$

We write $\Delta x_i = x_i - x_{i-1}$. Suppose f is a bounded real function defined on $[a,b]$, corresponding to each partition P of $[a,b]$ we put

$$M_i = \sup f(x) \qquad (x_{i-1} \le x \le x_i)$$

$$m_i = \inf f(x) \qquad (x_{i-1} \le x \le x_i)$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

and finally

$$\overline{\int_a^b} f dx = \inf U(P,f) \qquad \underline{\int_a^b} f dx = \sup L(P,f)$$

If the upper and lower integrals are equal, we say that f is *Riemann-integrable* on $[a,b]$. We write $f \in \mathscr{R}$ and $\mathscr{R}$ denotes the set of Riemann-integrable functions.

6.2 Definition Let α be a monotonically increasing function on $[a,b]$. Corresponding to each partition P of $[a,b]$, we write $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. For any real function f which is bounded on $[a,b]$, we put

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

where M_i and m_i have the same meaning in 6.1, and we define

$$\overline{\int_a^b} f d\alpha = \inf U(P,f,\alpha) \qquad \underline{\int_a^b} f d\alpha = \sup L(P,f,\alpha)$$

If these two values are equal, we denote their common value by $\int_a^b f d\alpha$, and this is the *Riemann-Stieltjes integral* of f with respect to α over $[a,b]$.

6.3 Definition We say that the partition P^* is a *refinement* of P if $P^* \supset P$ (that is, if every point of P is a point of P^*). Given two partition, P_1 and P_2, we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

6.4 Theorem If P^* is a refinement of P, then

$$L(P,f,\alpha) \le L(P^*,f,\alpha) \qquad U(P^*,f,\alpha) \le U(P,f,\alpha)$$

$$\underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha$$

6.8 Theorem If f is continuous on $[a, b]$ then $f \in \mathscr{R}(\alpha)$ on $[a, b]$. # R^k compact, f uniformly continuous. (4.19)

6.9 Theorem If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathscr{R}(\alpha)$. (We still assume, of course, that α is monotonic.)

6.10 Theorem Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.

6.11 Theorem Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$. # See 11.33

Properties of the Integral

If $f \in \mathscr{R}(\alpha)$, then $|f| \in \mathscr{R}(\alpha)$ and

$$\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| \, d\alpha$$

6.17 Theorem Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

Integration and Differentialtion

6.20 Theorem Let $f \in \mathscr{R}$ on $[a, b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t)dt$$

Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0 and $\underline{\underline{F'(x_0) = f(x_0)}}$.

6.21 The fundamental theorem of calculus If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

6.22 Theorem (integration by parts) Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

Integration of Vector-valued Functions

6.23 Definition Let $f_1, \cdots, f_k$ be real functions on $[a,b]$, and let $\mathbf{f} = (f_1, \cdots, f_k)$ be the corresponding mapping of $[a,b]$ into R^k. If α increases monotonically on $[a,b]$, to say that $\mathbf{f} \in \mathscr{R}(\alpha) means that \mathrm{f}_j \in \mathscr{R}(\alpha)$ for $j = 1, \cdots, k$. If this is the case, we define

$$\int_a^b \mathbf{f} d\alpha = \left(\int_a^b f_1 d\alpha, \cdots, \int_a^b f_k d\alpha \right)$$

Rectifiable Curves

6.26 Definition A continuous mapping γ of an interval $[a,b]$ into R^k is called a *curve* in R^k. If γ is one-to-one, γ is called an *arc*. If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve*.

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| \qquad \Lambda(\gamma) = \sup \Lambda(P, \gamma)$$

6.27 Theorem If γ' is continuous on $[a,b]$, then γ is *rectifiable*, and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt$$

7 Sequences and Series of Functions

Discussion of Main Problem

7.1 Definition Suppose $\{f_n\}$, $n = 1, 2, 3, \cdots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad x \in E$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the *limit*, or the *limit function*, of $\{f_n\}$. Sometimes we shall use a more descriptive terminology and shall say that "$\{f_n\}$ converges to f *pointwise* on E". Similaryly if $\sum f_n(x)$ converges for every $x \in E$, and if we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad x \in E$$

the function f is called the *sum* of the series $\sum f_n$.

Uniform Convergence

7.7 Definition We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, \cdots$, converges *uniformly* on E to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in E$.

It is clear that every uniformly convergent sequence is pointwise convergent. Quite explicitly, the difference between the two concepts is this:

- If $\{f_n\}$ converges pointwise on E, then there exists a function f such taht, for every $\epsilon > 0$, and for every $x \in E$, there is an integer N, depending on ϵ and on x.
- If $\{f_n\}$ converges uniformly on E, it is possible, for each $\epsilon > 0$, to find one integer N which will do for all $x \in E$.

7.8 Theorem The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E if and only if for every $\epsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies $|f_n(x) - f_m(x)| \leq \epsilon$.

7.9 Theorem Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ $(x \in E)$. Put $M_n = \sup_{x\in E} \left|f_n(x) - f_(x)\right|$. Then $f_n \to f$ *uniformly* on E if and only if $M_n \to 0$ as $n \to \infty$.

7.10 Theorem Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ $(x \in E, n = 1, 2, 3, \cdots)$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Uniform Convergence and Continuity !

7.11 Theorem Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that $\lim_{t\to x} f_n(t) = A_n$ $(n = 1, 2, 3, \cdots)$. Then $\{A_n\}$ converges, and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$$

In other words, the conclusion is that

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$$

7.12 Theorem If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Note: The converse is not true. A sequence of continuous functions may converge to a continuous function, although the convergence is not uniform.

7.13 Theorem Suppose K is compact, and

(a) $\{f_n\}$ is a sequence of continuous functions on K,

(b) $\{f_n\}$ converges pointwise to a continuous function f on K,

(c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \cdots$

Then $f_n \to f$ uniformly on K.

7.14 Definition If X is a metric space, $\mathscr{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X.

We associate with each $f \in \mathscr{C}(X)$ its *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|$$

If $h = f + g$, then

$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\| \qquad \Rightarrow \qquad \|f + g\| \leq \|f\| + \|g\|$$

If we define the distance between $f \in \mathscr{C}(X)$ and $g \in \mathscr{C}(X)$ to be $\|f - g\|$, it follows that 2.15 for a metric are satisfied. Thus we have made $\mathscr{C}(X)$ into a metric space.

Accordingly, closed subsets of $\mathscr{C}(X)$ are sometimes called *uniformly closed*, the closure of a set $\mathscr{A} \subset \mathscr{C}(X)$ is called its *uniform closure*, and so on ...

7.15 Theorem The above metric make $\mathscr{C}(X)$ into a *complete* metric space.

Uniform Convergence and Integration

7.16 Theorem Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \cdots$, and suppose $f_n \to f$ uniformly on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and

$$\int_a^b f d\alpha = \lim_{n \to \infty} \int_a^b f_n d\alpha$$

Uniform Convergence and Differentiation

7.17 Theorem Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on$[a, b]$, to a function f, and

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

7.18 Theorem There exists a real continuous function on the real line which is nowhere differentiable.

Equicontinuous Families of Functions

7.19 Definition Let $\{f_n\}$ be a sequence of functions defined on a set E.

We say that $\{f_n\}$ is *pointwise bounded* on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x)$$

We say that $\{f_n\}$ is *uniformly bounded* on E if there exists a number M such that

$$|f_n(x)| < M$$

However, even if $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, there need not exist a subsequence which converges pointwise on E.

7.22 Definition A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be *quicontinuous* on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta$, $x \in E$, $y \in E$, and $f \in \mathscr{F}$. Here d denotes the metric of X.

7.23 Theorem If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

7.24 Theorem If K is a compact metric space, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \cdots$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.

7.25 Theorem If K is compact, if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3, \cdots$, and if $\{f_n\}$ is *pointwise bounded and equicontinuous* on K, then

(a) $\{f_n\}$ is uniformly bounded on K,

(b) $\{f_n\}$ contains a uniformly convergent subsequence.

The Stone-Weierstrass Theorem

7.26 Theorem If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real. # This is the form in which the theorem was originally discovered by Weierstrass.

7.28 Definition A family $\mathscr{A}$ of complex functions defined on a set E is said to be an *algebra* if (i) $f + g \in \mathscr{A}$, (ii) $fg \in \mathscr{A}$, (iii) $cf \in \mathscr{A}$ for all $f \in \mathscr{A}$, $g \in \mathscr{A}$ and for all complex constants c, that is, if $\mathscr{A}$ is closed under addition, multiplication, and scalar multiplication.

If $\mathscr{A}$ has the property that $f \in \mathscr{A}$ whenever $f_n \in \mathscr{A}$ $(n = 1, 2, 3, \cdots)$ and $f_n \to f$ uniformly on E, then $\mathscr{A}$ is said to be *uniformly closed.*

Let $\mathscr{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$. Then $\mathscr{B}$ is called the *uniform closure* of $\mathscr{A}$. # See 7.14

7.29 Theorem Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.

7.30 Definition Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that $\mathscr{A}$ *vanishes at no point of* E.

7.31 Theorem Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that $f(x_1) = c_1$ and $f(x_2) = c_2$.

7.32 Theorem ! Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K. *Note*: 7.32 does not hold for complex algebras.

STEP 1 If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.

STEP 2 If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, then $\max(f, g) \in \mathscr{B}$ and $\min(f, g) \in \mathscr{B}$.

STEP 3 Given a real function f, continuous on K, a point $x \in K$, and $\epsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \epsilon$ $(t \in K)$.

STEP 4 Given a real function f, continuous on K, and $\epsilon > 0$, there exists a function $h \in \mathscr{B}$ such that $|h(x) - f(x)| < \epsilon$ $(x \in K)$.

7.33 Theorem Suppose $\mathscr{A}$ is a self-adjoint algebra of complex continuous functions on a compact set K, $\mathscr{A}$ separates points on K, and $\mathscr{A}$ vanishes at no point of K. Then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K. In other words, $\mathscr{A}$ is *dense* $\mathscr{C}(K)$.

8 Some Special Functions

Power Series

8.1 Theorem Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n$ ($|x| < R$). Then the series converges *uniformly* on $[-R+\epsilon, R-\epsilon]$, no matter which $\epsilon > 0$ is chosen. The function f is continuous and differentiable in $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$.

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n x^{n-k}$$

8.3 Theorem Given a double sequence $\{a_{ij}\}$, suppose that $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ and $\sum b_i$ converges. Then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

8.4 Theorem Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

the series converging in $|x| < R$. If $-R < a < R$, then f can be expanded in a power series about the point $x = a$ which converges in $|x-a| < R - |a|$, and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This is an extension of 5.15 and is also known as *Taylor's theorem*.

8.5 Theorem Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converge in the segment $S = (-R, R)$. Let E be the set of all $x \in S$ at which

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

If E has a limit point in S, then $a_n = b_n$ for $n = 0, 1, 2, \cdots$. Hence the equation holds for all $x \in S$.

The Exponential and Logarithmic Functions

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

The Trigonometric Functions

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)] \qquad S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$

The Algebraic Completeness of the Complex Field

8.8 Theorem Suppose $a_0, \cdots, a_n$ are complex numbers, $n \geq 1$, $a_n \neq 0$,

$$P(z) = \sum_{k=0}^{n} a_k z^k$$

Then $P(z) = 0$ for some complex number z.

Fourier Series

8.9 Definition A *trigonometric polynomial* is a finite sum of the norm (x real)

$$f(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) = \sum_{-N}^{N} c_n e^{inx}$$

where $a_0, \cdots, a_N$, $b_0, \cdots, b_N$ are complex numbers...

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 1 & \text{(if n=0)}, \\ 0 & \text{(if n=}\pm 1, \pm 2, \cdots). \end{cases}$$

Multiplied by e^{-imx}, where m is an integer, we have

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

We define a series of the form (x real)

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

If f is an integrable function on $[-\pi, \pi]$, the numbers c_m defined above for all integers m are called the *Fourier coefficients* of f, and this series formed with these coefficients is called the *Fourier series* of f.

8.10 Definition Let $\{\phi_n\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of complex functions on $[a, b]$ such that

$$\int_a^b \phi_n(x)\overline{\phi_m(x)} dx = 0 \qquad (n \neq m)$$

Then $\{\phi_n\}$ is said to be an *orthogonal system of functions* on $[a, b]$. If, in addition,

$$\int_a^b |\phi_n(x)|^2 \, dx = 1$$

for all n, $\{\phi_n\}$ is said to be *orthonormal*. For example, the functions $(2\pi)^{-1/2} e^{inx}$ form an orthonormal system on $[-\pi, \pi]$. So do the real functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots$$

8.11 Theorem Let $\{\phi_n\}$ be orthonormal on $[a,b]$. Let $s_n(x)$ be the nth partial sum of the Fourier series of f, and suppose $t_n(x)$

$$s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x) \qquad t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$$

Then we have the equation below as the equality holds if and only if $\gamma_m = c_m$ $(m = 1, \cdots, n)$.

$$\int_a^b |f - s_n|^2 \, dx \le \int_a^b |f - t_n|^2 \, dx$$

8.12 Theorem If $\{\phi_n\}$ is orthonormal on $[a,b]$, and if

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x)$$

then

$$\sum_{n=1}^{\infty} |c_n|^2 \le \int_a^b |f(x)|^2 \, dx$$

In particular,

$$\lim_{n\to\infty} c_n = 0$$

8.13 Trigonometric series

$$s_N(x) = s_N(f;x) = \sum_{-N}^{N} c_n e^{inx}$$

In order to obtain an expression for s_N, we introduce the *Dirichlet kernel*

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin\left[(N+\frac{1}{2})x\right]}{\sin(x/2)}$$

$$(e^{ix} - 1)D_N(x) = e^{i(N+1)x} - e^{-iNx}$$

$$s_N(f;x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt \; e^{inx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{-N}^{N} e^{in(x-t)} dt$$

mark

8.14 Theorem If for some x, there are constants $\delta > 0$ and $M < \infty$ such that $|f(x+t) - f(x)| \le M\,|t|$ for all $t \in (-\delta, \delta)$, then

$$\lim_{N\to\infty} s_N(f;x) = f(x)$$

8.15 Theorem If f is continuous (with period 2π) and if $\epsilon > 0$, then there is a trigonometric polynomial P such that $|P(x) - f(x)| < \epsilon$ for all real x.

8.16 Parseval's theorem Suppose f and g are Riemann-integrable functions with period 2π, and

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx} \qquad g(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{inx}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \sum_{-\infty}^{\infty} c_n \overline{\gamma_n} \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\,dx = \sum_{-\infty}^{\infty} |c_n|^2$$

The Gamma Function

8.17 Definition For $0 < x < \infty$,

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$$

8.18 Theorem

(a) The functional equation $\Gamma(x+1) = x\Gamma(x)$ holds if $0 < x < \infty$.

(b) $\Gamma(n+1) = n!$ for $n = 1, 2, 3, \cdots$

(c) $\log \Gamma$ is convex on $(0, \infty)$.

8.19 Theorem If f is a positive function on $(0, \infty)$ such that

(a) $f(x+1) = xf(x)$,

(b) $f(1) = 1$,

(c) $\log f$ is convex,

then $f(x) = \Gamma(x)$.

8.20 Theorem If $x > 0$ and $y > 0$, then

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

8.21 Theorem

$$2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right)$$

8.22 Stirling's formula This provides a simple approximate expression for $\Gamma(x+1)$ when x is large (hence for $n!$ when n is large). The formula is

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x\sqrt{2\pi x}} = 1$$

9 Functions of Several Variables

Linear Transformations

... ...

Differentiation

9.11 Definition Suppose E is an open set in R^n, $\mathbf{f}$ maps E into R^m, and $\mathbf{x} \in E$. If there exists a linear transformation A of R^n into R^m such that

$$\lim_{\mathbf{h}\to 0} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h}|}{|\mathbf{h}|} = 0$$

then we say that $\mathbf{f}$ is *differentiable* at $\mathbf{x}$, and we write

$$\mathbf{f}'(\mathbf{x}) = A$$

If $\mathbf{f}$ is differentiable at every $\mathbf{x} \in E$, we say that $\mathbf{f}$ is *differentiable in* E.

9.16 Partial derivatives We again consider a function $\mathbf{f}$ that maps an open set $E \subset R^n$ into R^m. Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ be the standard bases of R^n and R^m. The *components* of $\mathbf{f}$ are the real functions $f_1, \cdots, f_m$ defined by $(\mathbf{x} \in E)$

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x})\mathbf{u}_i$$

or, equivalently, by $f_i(x) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}_i$ $1 \le i \le m$.

For $\mathbf{x} \in E$, $1 \le i \le m$, $1 \le j \le n$, we define

$$(D_j f_i)(\mathbf{x}) = \lim_{t\to 0} \frac{f_i(\mathbf{x} + t\mathbf{e}_j) - f_i(\mathbf{x})}{t}$$

provided the limit exists. We see that $D_j f_i$ is the derivative of f_i with respect to x_j, keeping the other variables fixed. The notation

$$D_j f_i = \frac{\partial f_i}{\partial x_j}$$

is therefore often used, and $D_j f_i$ is called a *partial derivative.*

9.17 Theorem Suppose $\mathbf{f}$ maps an open set $E \subset R^n$ into R^m, and $\mathbf{f}$ is differentiable at a point $\mathbf{x} \in E$. Then the partial derivatives $(D_j f_i)(\mathbf{x})$ exist, and

$$\mathbf{f}'(\mathbf{x})\mathbf{e}_j = \sum_{i=1}^{m} (D_j f_i)(\mathbf{x})\mathbf{u}_i \qquad (1 \le j \le n)$$

where $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ are the standard bases of R^n and R^m.

$$\mathbf{f}'(\mathbf{x}) = \begin{bmatrix} (D_1 f_1)(\mathbf{x}) & \cdots & (D_n f_1)(\mathbf{x}) \\ \cdots & \cdots & \cdots \\ (D_1 f_m)(\mathbf{x}) & \cdots & (D_n f_m)(\mathbf{x}) \end{bmatrix}$$

9.20 Definition A differentiable mapping $\mathbf{f}$ of an open set $E \subset R^n$ into R^m is said to be *continuously differentiable* in E if $\mathbf{f}'$ is a continuous mapping of E into $L(R^n, R^m)$.

More explicitly, it is required that to every $\mathbf{x} \in E$ and to every $\epsilon > 0$ corresponds a $\delta > 0$ such that $\|\mathbf{f}'(\mathbf{y}) - \mathbf{f}'(\mathbf{x})\| < \epsilon$ if $\mathbf{y} \in E$ and $|\mathbf{x} - \mathbf{y}| < \delta$. We also say that $\mathbf{f}$ is a $\mathscr{C}'$-mapping, or that $\mathbf{f} \in \mathscr{C}'(E)$.

9.21 Theorem Suppose $\mathbf{f}$ maps an open set $E \subset R^n$ into R^m. Then $\mathbf{f} \in \mathscr{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

The Contraction Principle

9.22 Definition Let X be a metric space, with metric d. If φ maps X into X and if there is a number $c < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq c d(x, y)$$

for all $x, y \in X$, then φ is said to be a *contraction* of X into X.

9.23 Theorem If X is a complete metric space, and if φ is a contraction of X into X, then there exists one and only one $x \in X$ such that $\varphi(x) = x$.

The Inverse Function Theorem

9.24 Theorem Suppose $\mathbf{f}$ is a $\mathscr{C}'$-mapping of an open set $E \subset T^n$ into R^n, $\mathbf{f}'(\mathbf{a})$ is invertible for some $\mathbf{a} \in E$, and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then

(a) there exist open sets U and V in R^n such that $\mathbf{a} \in U$, $\mathbf{b} \in V$, $\mathbf{f}$ is one-to-one on U, and $\mathbf{f}(U) = V$;

(b) if $\mathbf{g}$ is the inverse of $\mathbf{f}$, defined in V by $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$ ($\mathbf{x} \in U$), then $\mathbf{g} \in \mathscr{C}'(V)$.

9.25 Theorem If $\mathbf{f}$ is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into R^n and if $\mathbf{f}'(\mathbf{x})$ is invertible for every $\mathbf{x} \in E$, then $\mathbf{f}(W)$ is an open subset of R^n for every open set $W \subset E$.

In other words, $\mathbf{f}$ is an *open mapping* of E into R^n.

The Implicit Function Theorem

Every $A \in L(R^{n+m}, R^n)$ can be split into two linear transformations A_x and A_y, defined by $A_x\mathbf{h} = A(\mathbf{h}, \mathbf{0})$, $A_y\mathbf{k} = A(\mathbf{0}, \mathbf{k})$ for any $\mathbf{h} \in R^n$, $\mathbf{k} \in R^m$. Then $A_x \in L(R^n)$, $A_y \in L(R^m, R^n)$, and $A(\mathbf{h}, \mathbf{k}) = A_x\mathbf{h} + A_y\mathbf{k}$.

9.27 Theorem If $A \in L(R^{n+m}, R^n)$ and if A_x is invertible, then there corresponds to every $\mathbf{k} \in R^m$ a unique $\mathbf{h} \in R^n$ such that $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$.

9.28 Theorem Let $\mathbf{f}$ be a $\mathscr{C}'$-mapping of an open set $E \subset R^{n+m}$ into R^n, such that $\mathbf{f}(\mathbf{a}, \mathbf{b}) = 0$ for some point $(\mathbf{a}, \mathbf{b}) \in E$.

Put $A = \mathbf{f}'(\mathbf{a}, \mathbf{b})$ and assume that A_x is invertible.

Then there exist open sets $U \subset R^{n+m}$ and $W \subset R^m$, with $(\mathbf{a}, \mathbf{b}) \in U$ and $\mathbf{b} \in W$, having the following property :

To every $\mathbf{y} \in W$ corresponds a unique $\mathbf{x}$ such that $(\mathbf{x}, \mathbf{y}) \in U$ and $\mathbf{f}(\mathbf{x}, \mathbf{y}) = 0$. If this $\mathbf{x}$ is defined to be $\mathbf{g}(\mathbf{y})$, then $\mathbf{g}$ is a $\mathscr{C}'$-mapping of W into R^n, $\mathbf{g}(\mathbf{b}) = \mathbf{a}$, $\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$ $(\mathbf{y} \in W)$ and $\mathbf{g}'(\mathbf{b}) = -(A_x)^{-1} A_y$.

The Rank Theorem ?

9.30 Definitions Suppose X and Y are vector spaces, and $A \in L(X, Y)$, as in *Def. 9.6*. The *null space* of A, $\mathscr{N}(A)$ is the set of all $\mathbf{x} \in X$ at which $A\mathbf{x} = \mathbf{0}$. It is clear that $\mathscr{N}(A)$ is a vector space in X.

Likewise, the *range* of A, $\mathscr{R}(A)$ is a vector space in Y.

The *rank* of A is defined to be the dimension of $\mathscr{R}(A)$.

Let X be a vector space. An operator $P \in L(X)$ is said to be a *projection* in X if $P^2 = P$.

More explicitly, the requirement is that $P(P\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in X$. In other words, P fixes every vector in its range $\mathscr{R}(P)$.

Here are some elementary properties of projections :

(a) If P is a projection in X, then every $\mathbf{x} \in X$ has a unique representation of the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1 \in \mathscr{R}(P)$, $\mathbf{x}_2 \in \mathscr{R}(P)$. ($\mathbf{x}_1 = P\mathbf{x}$ and $P\mathbf{x}_2 = \mathbf{0}$)

(b) If X is a finite-dimensional vector space and if X_1 is a vector space in X, then there is a projection P in X with $\mathscr{R}(P) = X_1$.

9.32 Theorem Suppose m, n, r are nonnegative integers, $m \geq r$, $n \geq r$, $\mathbf{F}$ is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into R^m, and $\mathbf{F}'(x)$ has rank r for every $\mathbf{x} \in E$.

Fix $\mathbf{a} \in E$, put $A = \mathbf{F}'(\mathbf{a})$, let Y_1 be the range of A, and let P be a projection in R^m whose range is Y_1. Let Y_2 be the null space of P.

Then there are open sets U and V in R^n, with $\mathbf{a} \in U$, $U \subset E$, and there is a 1-1 $\mathscr{C}'$-mapping $\mathbf{H}$ of V onto U (whose inverse is also of class $\mathscr{C}'$) such that

$$\mathbf{F}(\mathbf{H}(\mathbf{x})) = A\mathbf{x} + \phi(A\mathbf{x}) \qquad (\mathbf{x} \in V)$$

where ϕ is a $\mathscr{C}'$-mapping of the open set $A(V) \subset Y_1$ into Y_2.

Determinants

9.33 Definition If $(j_1, \cdots, j_n)$ is an ordered n-tuple of integers, define

$$s(j_1, \cdots, j_n) = \prod_{p<q} (j_q - j_p)$$

where $x = 1$ if $x > 0$, $x = -1$ if $x < 0$, $x = 0$ if $x = 0$.

Let $[A]$ be the matrix of a linear operator A on R^n, relative to the standard basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, with entries $a(i, j)$ in the ith row and jth column. The determinant of $[A]$ is defined to the the number

$$\det [A] = \sum s(j_1, \cdots, j_n) a(1, j_1) a(2, j_2) \cdots a(n, j_n)$$

Derivatives of Higher Order

9.39 Definition Suppose f is a real function defined in an open set $E \subset R^n$ with partial derivatives $D_1 f, \cdots, D_n f$. If the functions $D_j f$ are themselves differentiable, then the *second-order partial derivatives* of f are defined by

$$D_{ij} f = D_i D_j f \qquad (i, j = 1, \cdots, n)$$

If all these functions $D_{ij} f$ are continuous in E, we say that f is of class $\mathscr{C}''$ in E or that $f \in \mathscr{C}''(E)$.

9.41 Theorem Suppose f is defined in an open set $E \subset R^2$, suppose that $D_1 f$, $D_{21} f$ and $D_2 f$ exist at every point of E, and $D_{21} f$ is continuous at some point $(a, b) \in E$.

Then $D_{12} f$ exists at (a, b) and

$$(D_{12} f)(a, b) = (D_{21} f)(a, b)$$

Corollary $(D_{12} f) = (D_{21} f)$ if $f \in \mathscr{C}''(E)$.

Differentiation of Integrals

$$\frac{d}{dt} \int_a^b \phi(x, t) dx = \int_a^b \frac{\partial \phi}{\partial t}(x, t) dx$$

9.42 Theorem Suppose

(a) $\phi(x, t)$ is defined for $a \leq x \leq b$, $c \leq t \leq d$;

(b) α is an increasing function on $[a, b]$;

(c) $\phi(x, t) = \phi^t(x) \in \mathscr{R}(\alpha)$ for every $t \in [a, b]$;

(d) $c < s < d$, and to every $\epsilon > 0$ corresponds a $\delta > 0$ such that

$$|(D_2 \phi)(x, t) - (D_2 \phi)(x, s)| < \epsilon$$

for all $x \in [a, b]$ and for all $t \in (s - \delta, s + \delta)$.

Define

$$f(t) = \int_a^b \phi(x, t) d\alpha(x) \qquad (c \leq t \leq d)$$

Then $(D_2 \phi)^s \in \mathscr{R}(\alpha)$, $f'(s)$ exists, and

$$f'(s) = \int_a^b (D_2 \phi)(x, s) d\alpha(x)$$

10 Integration of Differential Forms

Integration

10.2 Theorem For every $f \in \mathscr{C}(I_k)$, $L(f) = L'(f)$.

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_k}^{b_k} f_k dx_k = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f_2 dx_2 = \int_{I^k} f = L(h_1 h_2 \cdots h_k) = L(h)$$
$$= \prod_{i=1}^{k}\int_{a_i}^{b_i} h_i(x_i)dx_i = \int_{a_i}^{b_i}\int_{a_1}^{b_1}\cdots\int_{a_{i-1}}^{b_{i-1}}\int_{a_{i+1}}^{b_{i+1}}\cdots\int_{a_j}^{b_j} f_j dx_j = L'(h)$$

10.3 Definition The *support* of a real or complex function f on R^k is the closure of the set of all points $\mathbf{x} \in R^k$ at which $f(\mathbf{x}) \neq 0$. If f is continuous function with compact support, let I^k be any k-cell which contains the support of f and define

$$\int_{R^k} f = \int_{I^k} f$$

The integral so defined is evidently independent of the choice of I^k, provided only that I^k contains the support of f.

Let Q^k be the k-simplex which consists of all points $\mathbf{x} = (x_1, \cdots, x_k)$ in R^k for which $x_1 + \cdots + x_k \leq 1$ and $x_i \geq 0$ for $i = 1, \cdots, k$.

Primitive Mappings

10.5 Definition If $\mathbf{G}$ maps an open set $E \subset R^n$ into R^n, and if there is an integer m and a real function g with domain E such that

$$\mathbf{G}(\mathbf{x}) = \sum_{i \neq m} x_i \mathbf{e}_i + g(\mathbf{x})\mathbf{e}_m \qquad (\mathbf{x} \in E)$$

then we call $\mathbf{G}$ *primitive*. A primitive mapping is thus one that changes at most one coordinate.

10.6 Definition A linear operator B on R^n that interchanges some pair of members of the standard basis and leaves the others fixed will be called a *flip*.

10.7 Theorem Suppose $\mathbf{F}$ is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into R^n, $\mathbf{0} \in E$, $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, and $\mathbf{F}'(\mathbf{0})$ is invertible.

Then there is a neighborhood of $\mathbf{0}$ in R^n in which a representation

$$\mathbf{F}(\mathbf{x}) = B_1 \cdots B_{n-1}\mathbf{G}_n \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

is valid. Each $\mathbf{G}_i$ is a primitive $\mathscr{C}'$-mapping in some neighborhood of $\mathbf{0}$; $\mathbf{G}_i(\mathbf{0}) = \mathbf{0}$, $\mathbf{G}_i'(\mathbf{0})$ is invertible, and each B_i is either a flip or the identity operator.

Partitions of Unity

10.8 Theorem Suppose K is a compact subset of R^n, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions $\psi_1, \cdots, \psi_s \in \mathscr{C}(R^n)$ such that

(a) $0 \le \psi_i \le 1$ for $1 \le i \le s$;

(b) each ψ_i has its support in some V_S;

(c) $\psi_1(\mathbf{x}) + \cdots + \psi_S(\mathbf{x}) = 1$ for every $\mathbf{x} \in K$.

Because of (c), $\{\psi_i\}$ is called a *partition of unity*, and (b) is sometimes expressed by saying that $\{\psi_i\}$ is *subordinate* to the cover $\{V_\alpha\}$.

Corollary If $f \in \mathscr{C}(R^n)$ and the support of f lies in K, then

$$f = \sum_{i=1}^{s} \psi_i f$$

Each $\psi_i f$ has its support in some V_α.

Change of Variables

10.9 Theorem Suppose T is a 1-1 $\mathscr{C}'$-mapping of an open set $E \subset R^k$ into R^k such that $J_T(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. If f is a continuous function on R^k whose support is compact and lies in $T(E)$, then

$$\int_{R^k} f(\mathbf{y}) d\mathbf{y} = \int_{R^k} f(T(\mathbf{x})) \left| J_T(\mathbf{x}) \right| d\mathbf{x}$$

Differential Forms

It is a curious feature of Stokes' theorem that the only thing that is difficult about it is the elaborate structure of definitions that are needed for its statements. These definitions concern differential forms, their derivatives, boundaries, and orientatoin.

10.10 Definition Suppose E is an open set in R^n. A k *surface* in E is a $\mathscr{C}'$-mapping Φ from a compact set $D \subset R^k$ into E.

D is called the *parameter domain* of Φ. Points of D will be denoted by $\mathbf{u} = (u_1, \cdots, u_k)$.

We stress that k-surface in E are defined to be *mapping* into E, not *subsets* of E.

10.11 Definition Suppose E is an open set in R^n. A *differential form of order* $k \ge 1$ in E (briefly, a k-form in E) is a function ω symbolically represented by the sum

$$\omega = \sum a_{i_1 \cdots i_k}(\mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

(the indices range independently from 1 to n), which assigns to each k-surface Φ in E a number $\omega(\Phi) = \int_\Phi \omega$, according to the rule

$$\int_\Phi \omega = \int_D \sum a_{i_1 \cdots i_k}(\Phi(\mathbf{u})) \frac{\partial(x_{i_1}, \cdots, x_{i_k})}{\partial(u_1, \cdots, u_k)} d\mathbf{u}$$

where D is the parameter domain of Φ.

10.13 Elementary properties Let ω, ω_1, ω_2 be k-forms in E. We write $\omega_1 = \omega_2$ if and only if $\omega_1(\Phi) = \omega_2(\Phi)$ for every k-surface *in*Φ in E.

$$\int_\Phi a\omega_1 + b\omega_2 = a \int_\Phi \omega_1 + b \int_\Phi \omega_2$$

10.14 Basic k-forms If $i_1, \cdots, i_k$ are integers such that $1 \le i_1 < i_2 < \cdots < i_k \le n$, and if I is the ordered k-tuple $\{i_1, \cdots, i_k\}$, then we call I an *increasing k-index*, and we use the brief notation

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

These forms dx_I are the so-called *basic k-forms* in R^n.

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = \epsilon(j_1, \cdots, j_k) dx_J = s(j_1, \cdots, j_k) dx_J$$

If every k-tuple is converted to an increasing k-index, then we obtain the so-called *standard presentation* of ω :

$$\omega = \sum_I b_I(\mathbf{x}) dx_I$$

10.15 Theorem Suppose

$$\omega = \sum_I b_I(\mathbf{x}) dx_I$$

is the standard presentation of a k-form ω in an open set $E \subset R^n$. If $\omega = 0$ in E, then $b_I(\mathbf{x}) = 0$ for every increasing k-index I and for every $\mathbf{x} \in E$.

10.16 Products of basic k-forms Suppose

$$I = \{i_1, \cdots i_p\} \qquad J = \{j_1, \cdots, j_q\}$$

where $1 \le i_1 < \cdots < i_p \le n$ and $1 \le j_1 < \cdots < j_p \le n$. The *product* of the corresponding basic forms dx_I and dx_J in R^n is a $(p+q)$-form in R^n, denoted by the symbol $dx_I \wedge dx_J$, and defined by

$$dx_I \wedge dx_J = dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q}$$

10.17 Multiplication Suppose ω and λ are p- and q-forms, respectively, in some open set $E \subset R^n$, with standard presentations

$$\omega = \sum_I b_I(\mathbf{x}) dx_I \qquad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

where I and J range over all increasing p-indices and over all increasing q-indices taken from the set $\{1, \cdots, n\}$. Their product, denoted by the symbol $\omega \wedge \lambda$, is defined to be

$$\omega \wedge \lambda = \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J$$

10.18 Differentiation We shall now define a differentiation operator d which associates a $(k+1)$-form $d\omega$ to each k-form ω of class $\mathscr{C}'$ in some open set $E \subset R^n$.

A 0-form of class $\mathscr{C}'$ in E is just a real function $f \in \mathscr{C}'(E)$, and we define

$$df = \sum_{i=1}^{n} (D_i f)(\mathbf{x}) dx_i$$

If $\omega = \sum b_I(\mathbf{x})dx_I$ is the standard presentation of a k-form ω, and $b_I \in \mathscr{C}'(E)$ for each increasing k-index I, then we define

$$d\omega = \sum_I (db_I) \wedge dx_I$$

10.20 Theorem

(a) If ω and λ are k- and m- forms, respectively, of class $\mathscr{C}'$ in E, then

$$d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda$$

.

(b) If ω is of class $\mathscr{C}''$ in E, then $d^2\omega = d(d\omega) = 0$.

10.21 Change of variables Suppose E is an open set in R^n, T is a $\mathscr{C}'$-mapping of E into an open set $V \subset R^m$, and ω is a k-form in V, whose standard presentation is (we use $\mathbf{y}$ for points of V, $\mathbf{x}$ for points of E.)

$$\omega = \sum_I b_I(\mathbf{y})dy_I$$

Let $t_1, \cdots, t_m$ be the components of T : if $\mathbf{y} = (y_1, \cdots, y_m) = T(\mathbf{x})$, then $y_i = t_i(\mathbf{x})$. According to 10.18,

$$dt_i = \sum_{j=1}^{n} (D_j t_i)(\mathbf{x})dx_j \qquad (1 \le i \le m)$$

Thus each dt_i is a 1-form in E.

The mapping T transforms ω into a k-form ω_T in E, whose definition is

$$\omega_T = \sum_I b_I(T(\mathbf{x}))dt_{i_1} \wedge \cdots \wedge dt_{i_k}$$

10.22 Theorem With E and T as in 10.21, let ω and λ be k- and m-forms in V, respectively. Then

(a) $(\omega + \lambda)_T = \omega_T + \lambda_T$ if $k = m$;

(b) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$;

(c) $d(\omega_T) = (d\omega)_T$ if ω is of class $\mathscr{C}'$ and T is of class $\mathscr{C}''$.

10.23 Theorem Suppose

$$\begin{aligned} &T : E \subset R^n \to V \subset R^m; \quad \omega : k\text{-form in } W; \\ &S : V \subset R^m \to W \subset R^p; \quad \omega_S : k\text{-form in } V; \quad \omega_{ST} : k\text{-form in } E; \end{aligned}$$

where ST is defined by $(ST)(\mathbf{x}) = S(T(\mathbf{x}))$. Then $(\omega_S)_T = \omega_{ST}$.

10.24 Theorem Suppose ω is a k-form in an open set $E \subset R^n$, Φ is a k-surface in E, with parameter domain $D \subset R^k$, and Δ is the k-surface in R^k, with parameter domain D, defined by $\Delta(\mathbf{u}) = \mathbf{u}(\mathbf{u} \in D)$. Then

$$\int_\Phi \omega = \int_\Delta \omega_\Phi$$

10.25 Theorem Suppose T is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, Φ is a k-surface in E, and ω is a k-form in V. Then

$$\int_{T\Phi} \omega = \int_{\Phi} \omega_T$$

Simplexes and Chains

10.26 Affine simplexes A mapping $\mathbf{f}$ that carries a vector space X into a vector space Y is said to be *affine* if $\mathbf{f} - \mathbf{f}(\mathbf{0})$ is linear. ($\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$ for some $A \in L(X, Y)$.)

An affine mapping of R^k into R^n is thus determined if we know $\mathbf{f}(\mathbf{0})$ and $\mathbf{f}(\mathbf{e}_i)$ for $1 \le i \le k$; as usual, $\{\mathbf{e}_1, \cdots, \mathbf{e}_k\}$ is the standard basis of R^k.

We **define** the *standard simplex* Q^k to be the set of all $\mathbf{u} \in R^k$ of the form

$$\mathbf{u} = \sum_{i=1}^{k} \alpha_i \mathbf{e}_i$$

such that $\alpha_i \ge 0$ for $i = 1, \cdots, k$ and $\sum \alpha_i \le 1$.

Assume now that $\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_k$ are points of R^n. The *oriented affine k-simplex*

$$\sigma = [\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_k]$$

is **defined** to be the k-surface in R^n with parameter domain Q^k which is given by the affine mapping

$$\sigma(\alpha_1 \mathbf{e}_1 + \cdots + \alpha_k \mathbf{e}_k) = \mathbf{p}_0 + \sum_{i=1}^{k} \alpha_i (\mathbf{p}_i - \mathbf{p}_0)$$

10.27 Theorem If σ is an oriented rectilinear k-simplex in an open set $E \subset R^n$ and if $\overline{\sigma} = \epsilon\sigma$ then

$$\int_{\overline{\sigma}} \omega = \epsilon \int_{\sigma} \omega$$

for every k-form ω in E.

10.28 Affine chains An *affine k-chain* Γ in an open set $E \subset R^n$ is a collection of finitely many oriented affine k-simplexes $\sigma_1, \cdots, \sigma_r$ in E. // These need not be distinct; a simplex may thus occur in Γ with a certain multiplicity.

If Γ is as above, and if ω is a k-form in E, we **define**

$$\int_{\Gamma} \omega = \sum_{i=1}^{r} \int_{\sigma_i} \omega$$

10.29 Boundaries For $k \ge 1$, the *boundary* of the oriented affine k-simplex

$$\sigma = [\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_k]$$

is defined to be the affine $(k-1)$-chain

$$\partial\sigma = \sum_{j=0}^{k} (-1)^j [\mathbf{p}_0, \cdots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \cdots \mathbf{p}_k]$$

10.30 Differentiable simplexes and cains Let T be a $\mathscr{C}''$-mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$; T need not be one-to-one. If σ is an oriented affine k-simplex in E, then the composite mapping $\Phi = T \circ \sigma$ (which we shall sometimes write in the simpler form $T\sigma$) is a k-surface in V, with parameter domain Q^k. We call Φ an *oriented k-simplex of class* $\mathscr{C}''$.

A finite collection Ψ of oriented k-simplexes $\Phi_1, \cdots, \Phi_r$ of class $\mathscr{C}''$ in V is called a *k-chain of class* $\mathscr{C}''$ in V. If ω is a k-form in V, we **define**

$$\int_\Psi \omega = \sum_{i=1}^{r} \int_{\Phi_i} \omega$$

and use the corresponding notation $\Psi = \sum \Phi_i$.

Finally, we **define** the boundary $\partial\Psi$ of the k-chain $\Psi = \sum \Phi_i$ to be the $(k-1)$-chain

$$\partial\Psi = \sum \partial\Phi_i$$

10.31 Let Q^n be the standard simplex in R^n, let σ_0 be the identity mapping with domain Q^n. (As in 10.26) σ_0 may be regarded as a positively oriented n-simplex in R^n Its boundary $\partial\sigma_0$ is an affine $(n-1)$-chain. This chain is called the *positively oriented boundary of the set* Q^n.

Stokes' Theorem

10.33 Theorem If Ψ is a k-chain of class $\mathscr{C}''$ in an open set $V \subset R^m$ and if ω is a $(k-1)$-form of class $\mathscr{C}'$ in V, then

$$\boxed{\int_\Psi d\omega = \int_{\partial\Psi} \omega}$$

Closed Forms and Exact Forms

Vector Analysis

11 The Lebesgue Theory

Set Functions

Construction of the Lebesgue Measure

Measure Spaces

Measurable Functions

Simple Functions

Integration

Comparison with the Riemann Integral

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Functions of Class L^2